The Frequency Distribution of Quantization Error in Digitizers for Coherent Sampling

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Abstract

In this paper the frequency distribution of quantization error in coherent sampling of digitizers will be investigated. As the distribution of the quantization error affects the values of THD and SNR, it is relevant for a correct interpretation of THD and SNR of sine waves in coherent sampling. The distribution of the quantization error will be analyzed as a function of N, total number of samples and M, number of cycles in the sample set and it will be proved that for certain cases, the quantization error falls on odd harmonics only of the frequency of the sampled sine wave.

Key words: Quantization error, Coherent sampling, THD, SNR.

Introduction

Quantization error is created when an analog signal is converted by a digitizer (ADC) to a digital form. In coherent sampling of sine waves, the ratio between the frequency of the sampled sine wave and the sampling frequency equals the ratio between N, the total number of samples and M, the number of sine wave cycles in the sample set. This guarantees an integer number of sine waves in the sample set.

The power of the quantization error is a function of the digitizer's resolution and its frequency distribution depends on the choice of N and M. The quantization error is unavoidable in the process of digitizing an analog signal and as the distribution of its power depends on the choice of N,M , the error will affect SNR and THD in different ways.

In an ideal ADC the only noise in the digitizer is the quantization error. If the amplitude of the quantization error has a uniform distribution in the range -0.5 LSB to +0.5 LSB, it can be shown that the SNR is given approximately by:

$$\begin{aligned} SNR(dB) &= 6.02 * j + 1.76 \\ where\ J &= number\ of\ bits \end{aligned} \quad (1)$$

In general, the distribution of the amplitude values of the quantization error depends on the sampling points in the decision levels and those in turn depend on the signal and the sampling rate. For a random signal the samples will be distributed uniformly along the decision levels resulting in a uniform distribution of the quantization error in the range -0.5LSB to +0.5 LSB. Theoretically, the quantization error can be eliminated for a ramp signal if the sampling points coincide with the boundary of quantiles (in which case the samples represent the exact values of the ramp at the sampled points). On the other hand, as the sine function is a nonlinear function, if the sampling rate is constant, the quantization error cannot be avoided. Still, if N,M (total number of samples, number of cycles) are relative primes, and the number of samples/cycle is large, the distribution of the values will be closely uniform, similar to the case of a random signal. The relation between the frequency bins corresponding to the quantization error and the frequency bins corresponding to the sampled sine wave and its harmonics will be analyzed including the effect on THD and SNR values for different choices of N and M.

The distribution of the quantization error in the frequency domain

As the Fourier transform is a linear transformation, a sampled sine wave containing only quantization error can be decomposed into a sequence representing a pure sine wave and a sequence representing the quantization error.

For a given sequence representing a sampled quantized sine wave with N samples and M cycles, there are three possible cases for the values of N,M:

Case 1: N,M are relative primes

In this case, all the samples representing the sine wave will correspond to different phase values coinciding with distinct points of a cycle. As a result, the samples will correspond closely to random positions in the various quantiles (decision levels of the ADC) resulting a quantization error sequence which is an aperiodic signal with its spectrum spread over all the frequency bins.This is illustrated in figure 1. In this case, the computed THD value will represent a 'true' value, as the bin numbers corresponding to the harmonics of the sine wave won't be affected by the quantization error. Also, as the quantization error is spread uniformly over all frequency bins, the noise component in the computation of the SNR will contain all the inherent noise in the digitizer as well as the quantization error. This will have an effect of degrading the SNR value.

Case 2: M divides N

In this case there are (N/M) samples/cycle and the quantization error forms a periodic sequence with a period equal to that of the sine wave. As a result, the discrete sequence representing the quantization error has a Fourier expansion with a fundamental frequency corresponding to bin # M and harmonics at bin #'s 2*M, 3*M,... up to the Nyquist frequency, with higher frequencies being aliased. Later it will be shown that for certain values of $(N'/M') = (N/M)/\gcd(N, M)$, the even harmonics of the frequency spectrum of the quantization error must by zero.

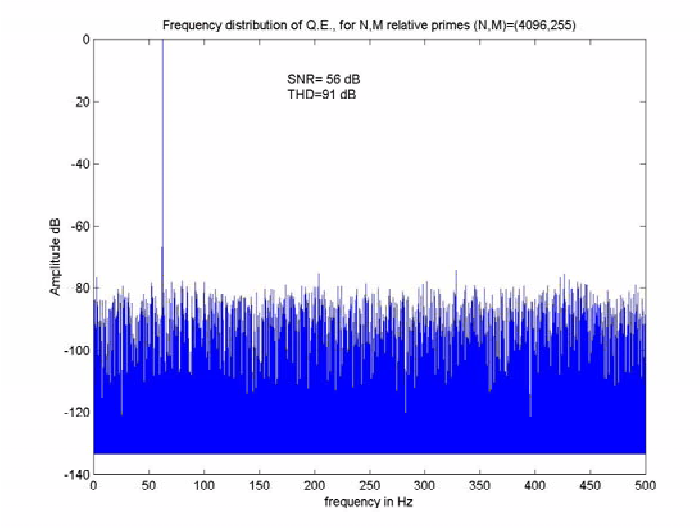

Figure 1 – Frequency Spectrum of a sampled sine wave with Q.E. for N,M relative Primes

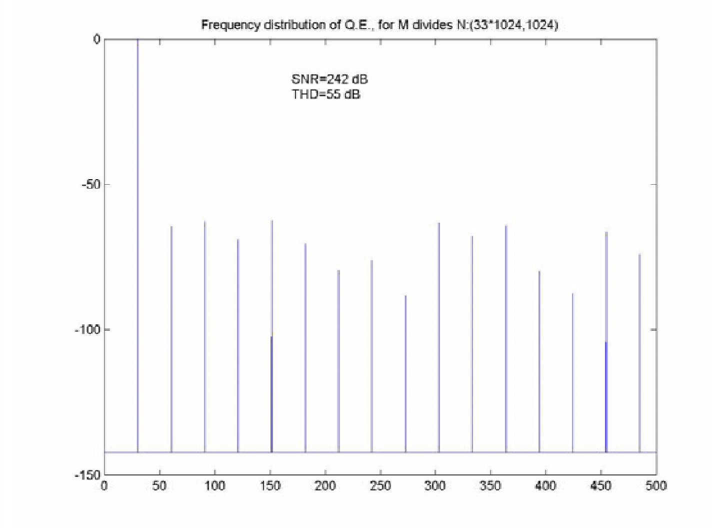

Figure 2 – Frequency spectrum of a sampled sine wave with Q.E. for M divides N

The case of M divides N, is illustrated in figure 2. The total energy of the quantization error is concentrated in the bins corresponding to the frequency of the sine wave and its harmonics. As a result, the quantization error will inflate the values of the harmonics of the sine wave and this in turn will degrade the value of THD. The SNR value, when compared to the previous case will improve since the quantization error is removed from nonhramonic bins to the harmonic bins.

Case 3: N,M have a common divisor>1 (e.g. 1024 and 96)

Let

$$(N', M') = (N, M)/\gcd(N, M) \qquad (2)$$

where gcd=greatest common divisor.

The sampled signal is periodic with period N' (where each period contains M' cycles of the sine wave) and therefore the quantization error forms a periodic sequence with a fundamental frequency corresponding to *bin number* $=gcd(N, M)$.

Since the quantization error is a periodic sequence, it has a Fourier expansion with harmonics at bin numbers: $2 * gcd(N, M), 3 * gcd(N, M), ...$(As in case 2, it will be shown later that for certain choices of N, M the even harmonics must be zero).

The frequency of the sine wave corresponds to *bin number* M, and since $M/M' = gcd(N, M)$, the frequency bin corresponding to the sine wave will coincide with *harmonic number* M' of the quantization error, or *bin number* $M' * gcd(N, M)$ of the quantization error. This is illustrated in figure 3.

The effect on THD and SNR in this case is the following: as the fundamental frequency of the sine wave coincides with one of the harmonics of the quantization error, and the harmonics of the sine wave coincide with a subset of the harmonics of the quantization error, the THD value will be degraded and the value of SNR will improve when compared to case 1, but not as much as in case 2.

Frequency domain analysis of the quantization error

Next we provide the mathematical analysis of the quantization error in the frequency domain for various choices of N, M and show that for certain choices, the quantization error falls on odd harmonics only.

In the previous section we found that if N, M are not relative primes, then the sequence of samples is periodic. Hence the sequence has a Fourier expansion with fundamental frequency at bin # = $\gcd(N, M)$ and the quantization error coincide with some fundamental frequency and its harmonics. The next lemma provides the conditions, in general, for a sequence representing an arbitrary periodic signal to have only odd harmonics in its Fourier expansion. Later the lemma will be applied to coherent sampling of sine waves to determine the cases for which the even harmonics must be zero.

Lemma

Let $x(n)$ $n = 0, 1, ..., N - 1$ be a sequence representing a sampled periodic signal, with N samples and M cycles such that N, M are not relative primes.

Let $N' = N/gcd(N, M)$ and $M' = M/gcd(N, M)$.

In the original sequence, there are $(N'\ samples)/(M'\ cycles)$ and the subsequence

$$x'(n)\ n = 0, .., N' - 1$$

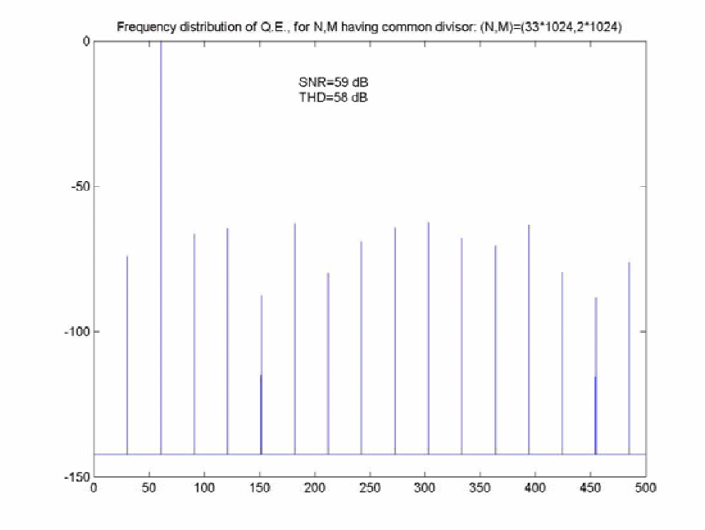

Figure 3 – Frequency distribution of a sampled sine wave with Q.E. for N,M having a common divisor

represents a period in the original sequence.

(1) If N' is even and $x'(n)$ satisfies the following condition:

$$x'(k) = -x'(k + N'/2) \quad k = 0, 1, ..., N'/2 - 1 \tag{3}$$

then the even harmonics in the Fourier expansion of the sequence $x(n)$ must be zero.

Proof

The DFT of $x(n)$ is given by:

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \qquad K = 0, 1, ..., N-1 \tag{4}$$

The bin number corresponding to the fundamental frequency of the periodic quantization noise equals to

$$bin(Q.E.) = gcd(N, M)$$

The bin number corresponding to the first even harmonic of $bin(Q.E.)\ is$:

$$2 * gcd(N, M) = 2 * M/M'$$

The DFT value corresponding to bin number $2 * M/M'$ is given by:

$$X(2 * M/M') = \tag{5}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(2*M/M')/N} =$$

$$\begin{aligned} = & \sum_{n=0}^{N'-1} x(n) e^{-j2\pi n(2*M/M')/N} + ... \\ & .. + \sum_{n=N'}^{2*N'-1} x(n) e^{-j2\pi n(2*M/M')/N} + ... \\ & .. + \sum_{n=N-(N'-1)}^{N} x(n) e^{-j2\pi n(2*M/M')/N} \end{aligned}$$

Now consider the first summation and rearrange the terms in pairs (separated by $N'/2$ terms):

$$\sum_{n=0}^{N'-1} x(n) e^{-j2\pi n(2*M/M')/N} = \tag{6}$$

$$\begin{aligned} = & \sum_{n=0}^{N'/2-1} [x(n) e^{-j2\pi n(2*M/M')/N} + .. \\ & .. + x(n + N'/2) e^{-j2\pi (n+N'/2)(2*M/M')/N}] \\ = & \sum_{n=0}^{N'/2-1} [x(n) e^{-j2\pi n(2*M/M')/N} + .. \\ & .. + x(n + N'/2) e^{-j2\pi n(2*M/M')/N} e^{-j2\pi (N' \cdot M)/(M'N)}] \end{aligned}$$

But

$$\begin{aligned} N &= \gcd(N, M) \cdot N' \\ M &= \gcd(N, M) \cdot M' \\ (N' \cdot M)/(M'N) &= \frac{N/\gcd(N, M) \cdot M}{M/\gcd(N, M) \cdot N} = 1 \\ \text{and so} \quad e^{-j2\pi (N' \cdot M)/(M'N)} &= 1 \end{aligned}$$

and since

$$x'(k) = -x'(k + N'/2) \quad k = 0, 1, ..., N'/2 - 1$$

then

$$\sum_{n=0}^{N'-1} x(n) e^{-j2\pi n(2*M/M')/N} = \tag{7}$$

$$\begin{aligned} = & \sum_{n=0}^{N'/2-1} [x(n) e^{-j2\pi n(2*M/M')/N} + \\ & +(-1) x(n) e^{-j2\pi n(2*M/M')/N} (1)] \\ = & \ 0 \end{aligned}$$

The other summations can also be shown to be zero and so the first even harmonic is zero:

$$X(2 * M/M') = \tag{8}$$

$$
\begin{aligned}
&= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(2*M/M')/N} = \\
&= \sum_{n=0}^{N'-1} x(n) e^{-j2\pi n(2*M/M')/N} + \\
&\quad + \sum_{n=N'}^{2*N'-1} x(n) e^{-j2\pi n(2*M/M')/N} + ... \\
&\quad ... + \sum_{n=N-(N'-1)}^{N} x(n) e^{-j2\pi n(2*M/M')/N} = 0
\end{aligned}
$$

Similarly it can be shown that all the other even harmonics of M/M' are zero:

$$X(4*M/M'),\ X(6*M/M'), ... = 0$$

and so only the Odd harmonics of M/M' may be non-zero.

□

Corollary

For coherent sampling of sine waves if (N', M')=(Even, Odd), it can be shown that sample number N'/2 has a phase value of π,and so $x(n)$ satisfies the condition:

$$x(k) = -x(k + N'/2) \quad k = 0, 1, ..., N'/2 - 1$$

hence, by the above lemma the even harmonics in the Fourier expansion of $x(n)$ must be zero.

□

Figures 4 and 5, illustrate the above result for coherent sampling of a simulated sine wave with quantization error.

Figure 4 shows the three sequences representing coherent sampling of sine waves for the cases:

$$(N', M') = (E, O), (O, E), (O, O).$$

Figure 5 provides the frequency spectrum for these three cases. As can be seen when N'=Even, the even harmonics must be zero while for N'=Odd, both the odd and even harmonics in the DFT may not be zero.

Note:

The proof of the lemma is based on rearranging the terms in the summation of the subsequence x'(n) to show that the sum of each pair in the summation is zero. This rearrangement of the terms cannot be applied directly to the original sequence for the following reason. If the original sequence satisfies (N, M)=(Even,Even), then the sum of a pair whose members are distant from each other by $N/2$ terms, may not be zero.

To see this we note:

$$X(2*M/M') = \tag{9}$$

$$
\begin{aligned}
&= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(2*M/M')/N} = \\
&= \sum_{n=0}^{N/2-1} [x(n) e^{-j2\pi n(2*M/M')/N} + .. \\
&\quad .. + x(n+N/2) e^{-j2\pi(n+N/2)(2*M/M')/N}] \\
&= \sum_{n=0}^{N/2-1} [x(n) e^{-j2\pi n(2*M/M')/N} + .. \\
&\quad .. + x(n+N/2) e^{-j2\pi n(2*M/M')/N} e^{-j2\pi(N/2)(2*M/M')/N}] \\
&= \sum_{n=0}^{N/2-1} [x(n) e^{-j2\pi n(2*M/M')/N} + .. \\
&\quad .. + x(n+N/2) e^{-j2\pi n(2*M/M')/N} e^{-j2\pi(M/M')}]
\end{aligned}
$$

Now for $(N, M) = (E, E)$

$$
\begin{aligned}
x(k) &= x(k + N/2) \quad k = 0, 1, ..., N/2 - 1 \\
e^{-j2\pi M/M'} &= 1 \ \ (\text{since } M/M' = \text{gcd}(N, M)\text{=integer})
\end{aligned}
$$

This implies

$$X(2*M/M') = \tag{10}$$

$$
\begin{aligned}
&= \sum_{n=0}^{N/2-1} [x(n) e^{-j2\pi n(2*M)/N} + x(n) e^{-j2\pi n(2*M)/N}(1)] \\
&= \sum_{n=0}^{N/2-1} [2x(n) e^{-j2\pi n(2*M)/N}]
\end{aligned}
$$

and in this form each term in the summation may not zero.

Summary:

The choice of N,M determines the distribution of the quantization error along the frequency bins in coherent sampling of sine waves. Since THD and SNR are functions of the energy distribution of the signal and the quantization error in the frequency domain, the choice of N,M will affect these parameters.

When N, M are relative primes, the quantization error is distributed uniformly along all the frequency bins. In this case its impact on the harmonics of the tested tone is negligible and in this respect the value for THD is not affected by the contribution of the quantization error. The SNR will be degraded, since the quantization error will be computed as noise.

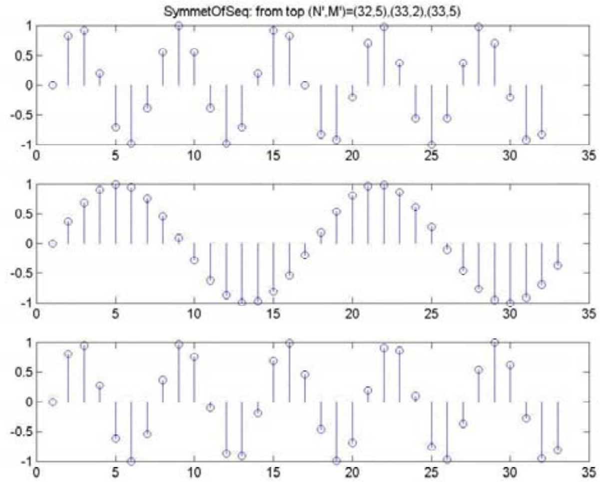

Figure 4 – Illustration of 3 subsequences with (from top): (N',M')=(E,O), (O,E), (O,O)

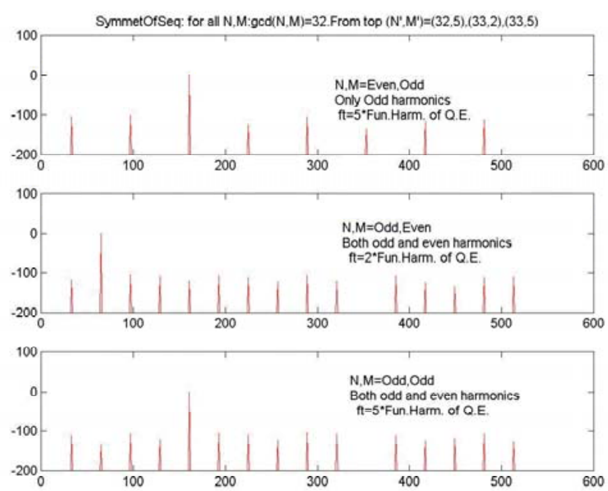

Figure 5 – frequency spectrum of the 3 subsequences from previous figure.

In the case that M divides N, the quantization error will coincide with the harmonics of the sampled sine wave (and if N is even, on the odd harmonics only). This in turn will have the effect of degrading the computed value of THD and improving the value for SNR when compared to the previous case.

When N, M have a common divisor>1, the bin corresponding to the tone frequency will coincide with one of the harmonics of the fundamental bin of the quantization error (specifically, harmonic number $M' = M/gcd(N, M)$). In this case, the tone and its harmonics are a subset of the fundamental and harmonics of the quantization error and therefore only a subset of the harmonics of the quantization error will be added to the harmonics of the tone, improving slightly the value of SNR and degrading slightly the value of THD when compared to the case where N, M are relative primes.

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